Spectral Representation of Self-Adjoint Problems for Layered Anisotropic Waveguides

Carlos R. Paiva and Afonso M. Barbosa, *Associate Member, IEEE*

Abstract —**Layered waveguides with lossless anisotropic layers in the polar configuration are analyzed through the unifying concept of a real self-adjoint operator. For a suitable definition of two-vector transverse eigenfunctions, general properties such as orthogonality and completeness relations are derived. The linear operator formalism is applied to closed waveguides inhomogeneously filled with anisotropic materials, including crystals and gyrotropic media. As an extension of the former theory to open waveguides, a grounded uniaxial dielectric slab with a coplanar optic axis is also analyzed: as for open isotropic waveguides, a complete spectral representation including the surface (proper eigenfunctions) as well as the pseudosurface modes (improper eigenfunctions) is presented.**

I. Introduction

The increasing use of anisotropic materials in applications ranging from microwaves and millimeter waves to optical frequencies has prompted the study of electromagnetic wave propagation in anisotropic media, particularly in connection with integrated circuits.

Several techniques have been developed to analyze layered anisotropic structures with direct application to optics and millimeter waves or as constituent parts of more complex structures in a building-block approach. Among the available techniques, the 4×4 matrix formalism appears to be especially well suited to handle general multilayered anisotropic or even bianisotropic media (e.g., [1]–[3]). However, very few fully analytical methods have been developed to study these problems: only for some anisotropic configurations is the analytical complexity not prohibitive (e.g., [4]–[7]), avoiding cumbersome calculations and lengthy expressions.

In a building-block approach it is necessary to describe the field components in each subregion in terms of a complete set of transverse mode functions. For isotropic dielectric structures the electromagnetic characterization of each subregion may be reduced to Sturm–Liouville systems [8]. However, when general anisotropic media have to be considered, the application of well-known results from the spectral theory of linear operators is usually required [9], [10].

Based on a self-adjoint operator formalism, a general analysis of closed multilayered waveguides containing gyromagnetic layers with polar configuration [4] has been presented by Mrozowski and Mazur [11]. For a suitable definition of two-vector transverse mode functions they have derived orthogonality and completeness relations. However, a similar linear-operator formalism suitable for dielectric planar waveguides containing uniaxial layers with coupled mode configurations has never been presented as far as the authors are aware.

In this paper, also based on a self-adjoint operator formalism, a general framework for the analysis of closed multilayered waveguides containing anisotropic layers with polar configuration is presented. Although the theoretical background of this framework is similar to [11], it is applicable to a more general class of anisotropic materials which includes crystals and gyrotropic (gyromagnetic or gyroelectric) media.

The previous linear operator formalism for closed waveguides is extended to open layered anisotropic waveguides where a complete spectral representation has to consider the surface and the pseudosurface modes [12]: taking into consideration a theorem related to the orthogonality of improper eigenfunctions of a linear operator by means of a perturbation approach [10], [13], a general analysis of a grounded uniaxial dielectric slab waveguide with a coplanar optical axis (polar configuration) is presented. Therefore, orthogonality relations for the surface (proper eigenfunctions) and pseudosurface modes (improper eigenfunctions) as well as a completeness relation for the spectral representation of the hybrid modes of this open anisotropic waveguide are presented for, to the best of the authors' knowledge, the first time. One should finally note that this analytical formulation is particularly relevant to the analysis of step discontinuities on uniaxial dielectric planar waveguides used in millimeter-wave and optical integrated circuits.

II. Eigenvalue Problems for Closed Waveguides

In this section, the general layered waveguide depicted in Fig. 1 will be analyzed. It is uniform in the y direction and is inhomogeneously filled with a spatially nondispersive, lossless, anisotropic medium. The waveguide is *closed* by electric and/or magnetic walls placed at $x = 0$ and $x = d$.

Time-harmonic variation of the form $\exp(j\omega t)$ will be considered. Hence, Maxwell's curl equations for source-free regions in the frequency domain are (boldface letters denoting vectors and boldface letters with an overbar denoting

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The authors are with the Departamento de Engenharia Electrotécnica e de Computadores and the Centro de Análise e Processamento de Sinais, Instituto Superior Técnico, Universidade Ténica de Lisboa, Av. Rovisco Pais, 1096 Lisboa Codex, Portugal.

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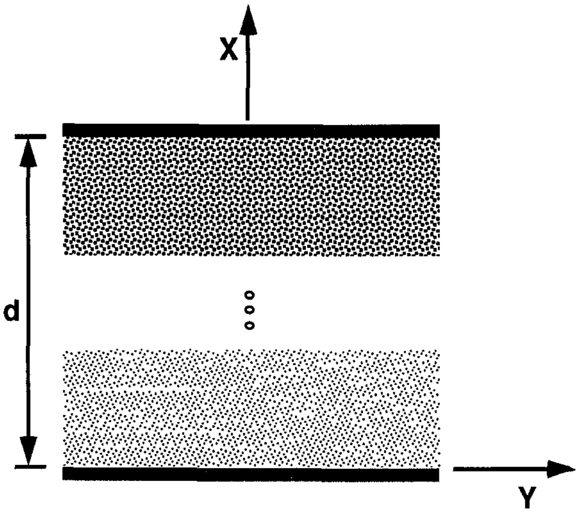


Fig. 1. Multilayered waveguide closed by electric and/or magnetic walls placed at $x=0$ and $x=d$. In general, $\bar{\epsilon}$ and $\bar{\mu}$ are piecewise-continuous functions of x for $x \in [0,d]$.

matrices and tensors)

$$\nabla \times \boldsymbol{H} = j\omega \boldsymbol{D}$$
$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B}. \tag{1}$$

Moreover, as z is the longitudinal direction and, since the structure is longitudinally uniform, the constitutive relations in the frequency domain are of the form

$$\boldsymbol{D}(\omega,x,z) = \epsilon_0 \bar{\epsilon}(\omega,x)\cdot \boldsymbol{E}(\omega,x,z)$$
$$\boldsymbol{B}(\omega,x,z) = \mu_0 \bar{\mu}(\omega,x)\cdot \boldsymbol{H}(\omega,x,z) \tag{2}$$

where $\bar{\epsilon}$ and $\bar{\mu}$ are, respectively, the relative dielectric permittivity and relative magnetic permeability tensors, which are considered piecewise-continuous functions of x for $0 \leqslant x \leqslant d$. As the medium is cold, i.e., spatially nondispersive, the constitutive relations are local.

Introducing normalized distances (e.g., $x' = k_0 x$, $z' = k_0 z$) as well as a normalized magnetic field,

$$\mathscr{H} = Z_0 \boldsymbol{H} \tag{3}$$

with $Z_0 = k_0/(\omega\epsilon_0) = (\omega\mu_0)/k_0$, then from (1)–(3) one obtains

$$\begin{bmatrix} \bar{\epsilon}(\omega,x') & j\nabla'\times \bar{I} \\ -j\nabla'\times\bar{I} & \bar{\mu}(\omega,x') \end{bmatrix}\cdot\begin{bmatrix} \boldsymbol{E} \\ \mathscr{H} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{4}$$

where $\bar{I}$ is the unit dyadic, i.e., $\bar{I} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$.

Electromagnetic fields have a z' dependence, $\exp(-j\beta z')$, with

$$\beta = \frac{k}{k_0} \tag{5}$$

where k is the longitudinal wavenumber. Therefore, one has $\nabla = k_0 \nabla'$ with

$$\nabla' = \partial_{x'}\hat{x} - j\beta\hat{z} \tag{6}$$

and

$$j\nabla'\times\bar{I} = \begin{bmatrix} 0 & -\beta & 0 \\ \beta & 0 & -j\partial_{x'} \\ 0 & j\partial_{x'} & 0 \end{bmatrix} \tag{7}$$

as $\partial/\partial y = 0$ since the waveguide is uniform in the y direction. The symbol $\partial_{x'}$ stands for $\partial/\partial x'$. Henceforth the factor $\exp[j(\omega t - \beta z')]$ will be omitted.

In this paper, only the polar configuration will be considered; i.e., $\bar{\epsilon}$ and $\bar{\mu}$ have the following matrix representation [4]:

$$\bar{\epsilon} = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & \epsilon_{yz} \\ 0 & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \qquad \bar{\mu} = \begin{bmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & \mu_{yz} \\ 0 & \mu_{zy} & \mu_{zz} \end{bmatrix} \tag{8}$$

in the (x,y,z) frame of Fig. 1. As the medium is lossless, $\bar{\epsilon}$ and $\bar{\mu}$ must be Hermitian [14], i.e.,

$$\bar{\epsilon}^{+} = \bar{\epsilon} \qquad \bar{\mu}^{+} = \bar{\mu} \tag{9}$$

where the superscript + means tranjugate or Hermitian conjugate.

In everything that follows, two kinds of lossless anisotropic materials will be considered: (i) crystals and (ii) gyrotropic media.

Crystals: If one disregards optical activity as well as the Faraday effect [15] $\bar{\epsilon}$, as well as $\bar{\mu}$ for the general case of magnetic crystals, is real, symmetric, and positive-definite [14]. Hence, all elements in (8) are positive and

$$\bar{\epsilon}^{T} = \bar{\epsilon} \qquad \bar{\mu}^{T} = \bar{\mu} \tag{10}$$

where the superscript T means transpose.

Gyrotropic Media: For a gyrotropic medium, one has from the Onsager relations [16] and for the polar configuration,

$$\epsilon_{zy} = -\epsilon_{yz} \qquad \mu_{zy} = -\mu_{yz} \tag{11}$$

since the applied magnetic field is aligned with the x axis. Hence, the off-diagonal elements in (8) must be imaginary, according to (9) and (11), while the diagonal elements are real. Therefore, for a gyrotropic medium, (8) may also be written in the form

$$\bar{\epsilon} = \begin{bmatrix} \epsilon_{\parallel} & 0 & 0 \\ 0 & \epsilon_{\perp} & j\epsilon_{x} \\ 0 & -j\epsilon_{x} & \epsilon_{\perp} \end{bmatrix} \qquad \bar{\mu} = \begin{bmatrix} \mu_{\parallel} & 0 & 0 \\ 0 & \mu_{\perp} & j\mu_{x} \\ 0 & -j\mu_{x} & \mu_{\perp} \end{bmatrix} \tag{12}$$

where $\epsilon_{yy} = \epsilon_{zz}$ and $\mu_{yy} = \mu_{zz}$ since $\bar{\epsilon}$ and $\bar{\mu}$ must be rotationally symmetric about x (the direction of the applied field).

After substituting (7) and (8) into (4), one obtains the two following coupled differential equations for E_y and $\mathscr{H}_y$:

$$\partial_{x'}\eta_1(x') + \frac{\nu_1(x')}{\epsilon_{zz}(x')} = \frac{\beta^2}{\mu_{xx}(x')}E_y$$
$$\partial_{x'}\eta_2(x') + \frac{\nu_2(x')}{\mu_{zz}(x')} = \frac{\beta^2}{\epsilon_{xx}(x')}\mathscr{H}_y \tag{13}$$

where

$$\eta_1(x') = \frac{1}{\mu_{zz}(x')}\left[\partial_{x'}E_y + j\mu_{zy}(x')\mathscr{H}_y\right]$$
$$\nu_1(x') = \Delta^{\epsilon}_{xx}(x')E_y - j\epsilon_{yz}(x')\partial_{x'}\mathscr{H}_y$$
$$\eta_2(x') = \frac{1}{\epsilon_{zz}(x')}\left[\partial_{x'}\mathscr{H}_y - j\epsilon_{zy}(x')E_y\right]$$
$$\nu_2(x') = \Delta^{\mu}_{xx}(x')\mathscr{H}_y + j\mu_{yz}(x')\partial_{x'}E_y \tag{14}$$

and where

$$\Delta^{\epsilon}_{xx} = \epsilon_{yy}\epsilon_{zz} - \epsilon_{yz}\epsilon_{zy}$$
$$\Delta^{\mu}_{xx} = \mu_{yy}\mu_{zz} - \mu_{yz}\mu_{zy} \tag{15}$$

are the cofactors of ϵ_{xx} and μ_{xx}, respectively, in (8).

The other field components may be expressed in terms of E_y and $\mathscr{H}_y$ as follows:

$$E_x = \frac{\beta}{\epsilon_{xx}(x')}\mathscr{H}_y$$
$$\mathscr{H}_x = -\frac{\beta}{\mu_{xx}(x')}E_y$$
$$E_z = -\frac{1}{\epsilon_{zz}(x')}\left[\epsilon_{zy}(x')E_y + j\partial_{x'}\mathscr{H}_y\right]$$
$$\mathscr{H}_z = \frac{1}{\mu_{zz}(x')}\left[j\partial_{x'}E_y - \mu_{zy}(x')\mathscr{H}_y\right]. \tag{16}$$

In order to write the coupled differential equations (13) as an eigenvalue equation, one has to define a two-vector eigenfunction for both crystals and gyrotropic media.

Crystals: Introducing

$$\phi_1 = E_y \qquad \phi_2 = j\mathscr{H}_y \tag{17}$$

and the two-vector transverse mode function

$$\mathbf{\Phi} = [\phi_1, \phi_2]^T \tag{18}$$

(13) and (14) may be recast in the form of an eigenvalue equation:

$$\overline{\mathscr{L}}\cdot\mathbf{\Phi} = \epsilon_{\text{eff}}\overline{\mathscr{W}}\cdot\mathbf{\Phi} \tag{19}$$

where $\overline{\mathscr{W}}$ is a real and symmetric "weight" operator given by

$$\overline{\mathscr{W}} = \begin{bmatrix} \dfrac{1}{\mu_{xx}(x')} & 0 \\ 0 & \dfrac{1}{\epsilon_{xx}(x')} \end{bmatrix} \tag{20}$$

and where the eigenvalue

$$\epsilon_{\text{eff}} = \beta^2 = \frac{k^2}{k_0^2} \tag{21}$$

represents an effective dielectric permittivity corresponding to the eigenfunction $\mathbf{\Phi}$. The linear operator $\overline{\mathscr{L}}$ is the 2×2 matrix differential operator

$$\overline{\mathscr{L}} = \begin{bmatrix} \mathscr{L}_{11} & \mathscr{L}_{12} \\ \mathscr{L}_{21} & \mathscr{L}_{22} \end{bmatrix} \tag{22}$$

with

$$\mathscr{L}_{11} = \partial_{x'}\frac{1}{\mu_{zz}(x')}\partial_{x'} + a_1(x')$$
$$\mathscr{L}_{12} = -b_1(x')\partial_{x'} + \partial_{x'}b_2(x')$$
$$\mathscr{L}_{21} = \partial_{x'}b_1(x') - b_2(x')\partial_{x'}$$
$$\mathscr{L}_{22} = \partial_{x'}\frac{1}{\epsilon_{zz}(x')}\partial_{x'} + a_2(x'). \tag{23}$$

For magnetic crystals, one has

$$a_1 = \frac{\Delta^{\epsilon}_{xx}}{\epsilon_{zz}} \qquad a_2 = \frac{\Delta^{\mu}_{xx}}{\mu_{zz}}$$
$$b_1 = \frac{\epsilon_{yz}}{\epsilon_{zz}} \qquad b_2 = \frac{\mu_{yz}}{\mu_{zz}}. \tag{24}$$

Gyrotropic Media: For a gyrotropic medium, if one introduces, instead of (17),

$$\phi_1 = E_y \qquad \phi_2 = \mathscr{H}_y \tag{25}$$

then (18)–(23) are still valid provided that

$$a_1 = \frac{\epsilon_\perp^2 - \epsilon_x^2}{\epsilon_\perp} \qquad a_2 = \frac{\mu_\perp^2 - \mu_x^2}{\mu_\perp}$$
$$b_1 = -\frac{\epsilon_x}{\epsilon_\perp} \qquad b_2 = \frac{\mu_x}{\mu_\perp}. \tag{26}$$

For gyromagnetic media, one has $\epsilon_x = 0$ and $\epsilon_\parallel = \epsilon_\perp = \epsilon_r$; then $a_1 = \epsilon_r(x')$, $b_1 \equiv 0$, and the problem is reduced to the one already treated by Mrozowski and Mazur in [11].

One should note that definitions (17) and (25) were chosen in order to obtain real operator $\overline{\mathscr{L}}$ for both cases.

The waveguide in Fig. 1 is closed at $x = 0$ and $x = d$ by electric or magnetic walls. Therefore, at $x' = 0$, d', the boundary conditions, according to (16), are (i) $E_y = \partial_{x'}\mathscr{H}_y = 0$ for an electric wall and (ii) $\mathscr{H}_y = \partial_{x'}E_y = 0$ for a magnetic wall. Hence, when the off-diagonal elements in (8) are null for $0 \leqslant x \leqslant d$, (19) yields two separate Sturm-Liouville systems [8], thus allowing the propagation of TE and TM modes.

As ϕ_1 and ϕ_2, defined in (17) or (25), have finite energy, they belong to the complete vector space Ω_0 of all real-valued functions which are Lebesgue square integrable over $[0, d']$, i.e., for $0 \leqslant x' \leqslant d'$. The domain of operator $\overline{\mathscr{W}}$, $\mathrm{D}(\overline{\mathscr{W}})$, will be the set of functions $\mathbf{\Phi} = [\phi_1, \phi_2]^T$ with $\phi_1, \phi_2 \in \Omega_0$ such that, at $x' = 0$, d', (i) $\phi_1 = \partial_{x'}\phi_2 = 0$ for an electric wall and (ii) $\phi_2 = \partial_{x'}\phi_1 = 0$ for a magnetic wall. Since all the expressions to which operator $\partial_{x'}$ is applied have to be continuous, the domain of $\overline{\mathscr{L}}$ is $\mathrm{D}(\overline{\mathscr{L}})$ with $\mathrm{D}(\overline{\mathscr{L}}) \subset \mathrm{D}(\overline{\mathscr{W}})$ such that

(a) ϕ_1

(b) ϕ_2

(c) $\dfrac{1}{\mu_{zz}(x')}\partial_{x'}\phi_1 + b_2(x')\phi_2$

(d) $\dfrac{1}{\epsilon_{zz}(x')}\partial_{x'}\phi_2 + b_1(x')\phi_1$

are continuous on the interval $[0, d']$. One can easily see that the continuity of functions (a)–(d) is equivalent to the continuity of the field components that are perpendicular to the x axis.

Introducing the following definition of inner product of the two elements $\boldsymbol{u} = [u_1, u_2]^T$ and $\boldsymbol{v} = [v_1, v_2]^T$ from $\mathrm{D}(\overline{\mathscr{L}})$ as

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \int_0^{d'}(u_1v_1 + u_2v_2)\,dx' \tag{27}$$

one can prove that $\overline{\mathscr{L}}$ is self-adjoint since it is symmetric [13] (Appendix I). Therefore, as $\overline{\mathscr{L}}$ is real, the eigenvalues ϵ_{eff_n} are real and the eigenfunctions $\mathbf{\Phi}_n$ can be chosen to be real

[13]. Moreover, the following orthogonality relation holds for any $\mathbf{\Phi}_m$ and $\mathbf{\Phi}_n$ from D($\bar{\mathscr{L}}$):

$$(\epsilon_{\mathrm{eff}_m}-\epsilon_{\mathrm{eff}_n})\langle\mathbf{\Phi}_m,\bar{\mathscr{W}}\cdot\mathbf{\Phi}_n\rangle=0. \tag{28}$$

With an appropriate normalization, one can write

$$\langle\mathbf{\Phi}_m,\bar{\mathscr{W}}\cdot\mathbf{\Phi}_n\rangle=\delta_{mn} \tag{29}$$

where δ_{mn} is the Kronecker delta. In fact, for $m\neq n$ one has $\epsilon_{\mathrm{eff}_m}\neq\epsilon_{\mathrm{eff}_n}$ as long as $b_1(x')$ and $b_2(x')$ in (23) are not identically null for $0\leqslant x'\leqslant d'$.

As the set of eigenfunctions $\{\mathbf{\Phi}_n\}$ span D($\bar{\mathscr{L}}$) [13], then, if $\mathbf{\Phi}\in$ D($\bar{\mathscr{L}}$),

$$\mathbf{\Phi}=\sum_{n=1}^{\infty}\alpha_n\mathbf{\Phi}_n \tag{30}$$

with

$$\alpha_n=\langle\mathbf{\Phi}_n,\bar{\mathscr{W}}\cdot\mathbf{\Phi}\rangle \tag{31}$$

according to (29). Hence, the following completeness relation is also valid:

$$\|\mathbf{\Phi}\|^2=\langle\mathbf{\Phi},\bar{\mathscr{W}}\cdot\mathbf{\Phi}\rangle=\sum_{n=1}^{\infty}\alpha_n^2. \tag{32}$$

III. Homogeneous Layers

In this section the special case of homogeneous layers will be considered. In fact, they frequently appear as models for constituent layers of multilayered structures used both in closed and open waveguides.

Inside a layer where $\bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\mu}}$ do not depend on x', one may write, according to (18)–(23),

$$\begin{aligned}\partial_{x'}^2\phi_1+p_1\phi_1&=q_1\partial_{x'}\phi_2\\ \partial_{x'}^2\phi_2+p_2\phi_2&=q_2\partial_{x'}\phi_1\end{aligned} \tag{33}$$

with

$$\begin{aligned}p_1&=\mu_{zz}\left(a_1-\frac{\epsilon_{\mathrm{eff}}}{\mu_{xx}}\right)\\ p_2&=\epsilon_{zz}\left(a_2-\frac{\epsilon_{\mathrm{eff}}}{\epsilon_{xx}}\right)\\ q_1&=\mu_{zz}(b_1-b_2)\\ q_2&=-\epsilon_{zz}(b_1-b_2).\end{aligned} \tag{34}$$

If $b_1=b_2$ then $q_1=q_2=0$. Hence, according to (33), ϕ_1 and ϕ_2 remain uncoupled as in the isotropic case. Any case in which $b_1=b_2$ will be disregarded in this paper.

From (33) and if $p_1,p_2,q_1,q_2\neq 0$ one obtains

$$\{\partial_{x'}^4+(p_1+p_2-q_1q_2)\partial_{x'}^2+p_1p_2\}\phi_i=0 \tag{35}$$

with $i=1,2$. To solve (35) one has to first solve the auxiliary biquadratic equation

$$h^4-\beta_2h^2+\beta_0=0 \tag{36}$$

where h is a transverse wavenumber. Hence, there are four transverse wavenumbers: $\pm h_a$ and $\pm h_b$, such that

$$\begin{aligned}\beta_0&=p_1p_2=h_a^2h_b^2\\ \beta_2&=p_1+p_2-q_1q_2=h_a^2+h_b^2.\end{aligned} \tag{37}$$

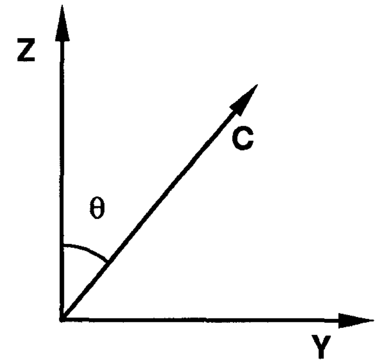


Fig. 2. Uniaxial polar orientation (C is the optical or crystal axis).

From (35) and (37) one may write

$$\begin{aligned}\phi_1&=\phi_a+\phi_b\\ \phi_2&=\xi_a\partial_{x'}\phi_a+\xi_b\partial_{x'}\phi_b\end{aligned} \tag{38}$$

where, for a layer with finite thickness,

$$\phi_s=A_s[\sin(h_sx')+\chi_s\cos(h_sx')] \tag{39}$$

with $s=a,b$ and where ξ_s are coupling coefficients. According to (33), these coupling coefficients are given by ($s=a,b$)

$$\xi_s=\frac{h_s^2-p_1}{h_s^2q_1}=\frac{q_2}{p_2-h_s^2}. \tag{40}$$

One should also note that if $p_1=0$ or $p_2=0$ (with $b_1\neq b_2$), a slightly different analysis should be developed since (35) is no longer valid (Appendix II).

Next, some examples of homogeneous layers with different types of anisotropy will be considered.

If the layer is a cold magnetoplasma with the externally applied constant magnetic field aligned with the x axis, then $\bar{\boldsymbol{\mu}}=\bar{\boldsymbol{I}}$ and $\bar{\boldsymbol{\epsilon}}$ is given by (12). Hence, β_0 and β_2 in (37) may be written as

$$\begin{aligned}\beta_0&=\frac{\epsilon_\perp}{\epsilon_\|}(\epsilon_\|-\epsilon_{\mathrm{eff}})(\epsilon_\delta-\epsilon_{\mathrm{eff}})\\ \beta_2&=2\epsilon_\perp-\left(1+\frac{\epsilon_\perp}{\epsilon_\|}\right)\epsilon_{\mathrm{eff}}\end{aligned} \tag{41}$$

where $\epsilon_\delta=(\epsilon_\perp^2-\epsilon_x^2)/\epsilon_\perp$.

Similarly, if the layer under consideration is a ferrite with an externally applied constant magnetic field aligned with the x axis, then $\bar{\boldsymbol{\epsilon}}=\epsilon_r\bar{\boldsymbol{I}}$ and $\bar{\boldsymbol{\mu}}$ is given by (12) with $\mu_\|=1$. Hence,

$$\begin{aligned}\beta_0&=\mu_\perp(\epsilon_r-\epsilon_{\mathrm{eff}})(\epsilon_r\mu_\delta-\epsilon_{\mathrm{eff}})\\ \beta_2&=2\epsilon_r\mu_\perp-(1+\mu_\perp)\epsilon_{\mathrm{eff}}\end{aligned} \tag{42}$$

where $\mu_\delta=(\mu_\perp^2-\mu_x^2)/\mu_\perp$.

Finally, if the layer is a nonmagnetic uniaxial crystal, then $\bar{\boldsymbol{\mu}}=\bar{\boldsymbol{I}}$ and [14]

$$\bar{\boldsymbol{\epsilon}}=\epsilon_\perp\bar{\boldsymbol{I}}+(\epsilon_\|-\epsilon_\perp)\hat{c}\hat{c} \tag{43}$$

where $\hat{c}$ is the unit eigenvector of $\bar{\boldsymbol{\epsilon}}$ corresponding to its nonrepeated eigenvalue $\epsilon_\|$; i.e., $\hat{c}$ is aligned with the optical axis. When $\hat{c}$ lies on the yz plane, one has (Fig. 2)

$$\hat{c}=\sin\theta\,\hat{y}+\cos\theta\,\hat{z} \tag{44}$$

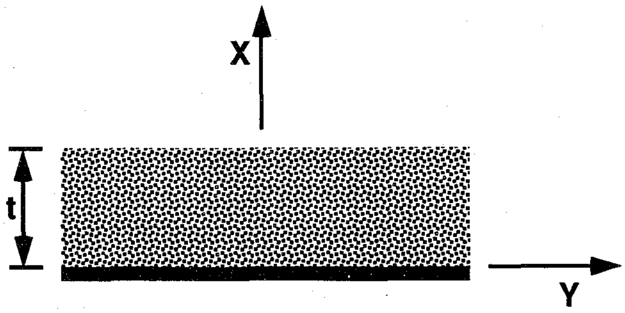


Fig. 3. Grounded dielectric slab. The slab is a uniaxial crystal with thickness t and a coplanar optical axis (Fig. 2).

and the anisotropy is of the polar type. In fact,

$$\bar{\epsilon} = \epsilon_\perp \hat{x}\hat{x} + \epsilon_{yy}\hat{y}\hat{y} + \epsilon_{zz}\hat{z}\hat{z} + \epsilon_{yz}(\hat{y}\hat{z} + \hat{z}\hat{y}) \quad (45)$$

where, according to (43) and (44),

$$\begin{aligned} \epsilon_{yy} &= \epsilon_\| \sin^2\theta + \epsilon_\perp \cos^2\theta \\ \epsilon_{zz} &= \epsilon_\| \cos^2\theta + \epsilon_\perp \sin^2\theta \\ \epsilon_{yz} &= (\epsilon_\| - \epsilon_\perp)\sin\theta\cos\theta. \end{aligned} \quad (46)$$

For this anisotropic case, ϕ_a and ϕ_b in (38) correspond to coupled ordinary and extraordinary field components of the hybrid wave (which are uncoupled for $\theta = 0, \pi/2$). Therefore, the subscripts $s = a, b$ will be replaced with $s = o, e$. Consequently one has, from (36) and (40),

$$h_o^2 = \epsilon_\perp - \epsilon_{\text{eff}} \qquad h_e^2 = \epsilon_\| - \frac{\epsilon_{zz}}{\epsilon_\perp}\epsilon_{\text{eff}}$$

$$\xi_o = -\frac{\epsilon_\perp}{h_0^2}\tan\theta \qquad \xi_e = \cot\theta. \quad (47)$$

IV. Grounded Uniaxial Dielectric Slab

In Section II multilayered *closed* anisotropic structures were studied. As an extension of that theory, an *open* structure will be analyzed in this section.

When the eigenvalue equation (19) is applied to *open* waveguides, the theory of Section II is no longer valid. In fact, for open waveguides, the differential operator $\bar{\mathscr{L}}$ is defined over an infinite (or semi-infinite) interval and has a discrete spectrum as well as a continuous spectrum.

In this section, the open waveguide depicted in Fig. 3 will be analyzed. It is a conductor-backed, x-cut, nonmagnetic uniaxial crystal with a coplanar optical axis (polar configuration—Fig. 2). The semi-infinite upper layer is the air; hence, for $t' < x'$, $\bar{\epsilon}(x') = \bar{I}$. The discrete spectrum of this slab waveguide has previously been analyzed by the authors [7], although this was on the basis of a different method which is unable to establish the orthogonality and completeness relations for the transverse eigenfunctions. The analysis presented in this section is useful since (i) it illustrates the application of eigenequation (19) to an open waveguide; (ii) the results concerning the discrete spectrum can be compared with those obtained in [7] by a different approach; (iii) a complete spectral representation including the discrete and the continuous spectra is presented; and (iv) orthogonality and completeness relations are given.

The surface modes constitute the discrete spectrum of $\bar{\mathscr{L}}$ and they also define its domain $D(\bar{\mathscr{L}})$ as the set of functions $\Phi = [\phi_1, \phi_2]^T$ with $\phi_1, \phi_2 \in \Omega_1$ such that

$$\phi_1(x'=0) = \partial_{x'}\phi_2(x'=0) = 0 \quad (48)$$

owing to the perfectly conducting plate at $x' = 0$ (Fig. 3) and where the continuity of functions (a)–(d) in Section II at $x' = t'$ is observed; Ω_1 is the vector space of square integrable functions over $[0, \infty[$, i.e., for $0 \leqslant x' < \infty$.

Then, according to (38) and (39), one should have, for the *hybrid surface modes* (with $\epsilon_{yz} \neq 0$),

$$\begin{aligned} \phi_1 &= A_o \sin(h_o x') + A_e \sin(h_e x') \\ \phi_2 &= \xi_o h_o A_o \cos(h_o x') + \xi_e h_e A_e \cos(h_e x') \end{aligned} \quad (49)$$

if $0 < x' < t'$ ($\chi_o = \chi_e = 0$) and where h_s and ξ_s ($s = o, e$) are given in (47). If $t' < x'$, one should have

$$\begin{aligned} \phi_1 &= B\exp[-\alpha_a(x'-t')] \\ \phi_2 &= C\exp[-\alpha_a(x'-t')] \end{aligned} \quad (50)$$

with

$$\alpha_a^2 = \epsilon_{\text{eff}} - 1. \quad (51)$$

In order to satisfy the radiation condition, one must have $\alpha_a > 0$; i.e., the surface modes are slow modes ($\epsilon_{\text{eff}} > 1$). One should note that, taking (17) into consideration, the other field components are easily derived according to (16): one just has to make $\epsilon_{xx} = \epsilon_\perp$, $\mu_{xx} = \mu_{zz} = 1$, and $\mu_{zy} = 0$, whereas $\epsilon_{zy} = \epsilon_{yz}$ and ϵ_{zz} are given in (46).

Imposing the continuity of (a)–(d) of Section II at $x' = t'$ and noting that ($s = o, e$),

$$\frac{\epsilon_{yz} - h_s^2\xi_s}{\epsilon_{zz}} = -\frac{h_o^2}{\epsilon_\perp}\xi_s \quad (52)$$

one gets

$$\begin{bmatrix} \gamma_o & \gamma_e \\ \xi_o\delta_o & \xi_e\delta_e \end{bmatrix} \cdot \begin{bmatrix} A_o \\ A_e \end{bmatrix} = \mathbf{0} \quad (53)$$

with

$$\begin{aligned} \gamma_s &= h_s\cos(h_s t') + \alpha_a \sin(h_s t') \\ \delta_s &= \epsilon_\perp h_s \alpha_a \cos(h_s t') - h_o^2 \sin(h_s t'). \end{aligned} \quad (54)$$

To obtain nontrivial solutions for (53), one has to ensure that

$$\xi_e\gamma_o\delta_e - \xi_o\gamma_e\delta_o = 0. \quad (55)$$

Hence, (55) is the modal equation for the surface modes of the uniaxial dielectric slab in Fig. 3 when $\epsilon_{yz} \neq 0$.

Remarking that $\xi_s = \epsilon_\perp \tau_s / h_o^2$, where τ_s are the coupling coefficients introduced in [7], one can easily see that [7, eq. (56)] is equivalent to (55).

The numerical solution of (55) shows some interesting results. The variation of ϵ_{eff} with θ for the propagating hybrid surface modes of the slab in Fig. 3 (the optical axis orientation was defined in Fig. 2) with $t = 0.15\lambda$ is presented in Fig. 4 for two different cases: (i) when the slab is a positive uniaxial crystal with $\epsilon_\| = 11.6$ and $\epsilon_\perp = 9.4$ (A curves, sapphire) there are two propagating modes; (ii) when the slab is a negative uniaxial crystal with $\epsilon_\| = 3.40$ and $\epsilon_\perp = 5.12$ (B curves, pyrolitic boron nitride) there are only two propagating modes for $\theta < \theta_0$ ($\theta_0 = 63.6°$) since for $\theta > \theta_0$ the second mode is at cutoff ($\epsilon_{\text{eff}} = 1$ for $\theta = \theta_0$) and only the fundamental mode is propagating. This last effect clearly reveals the importance of taking into account the anisotropy in the

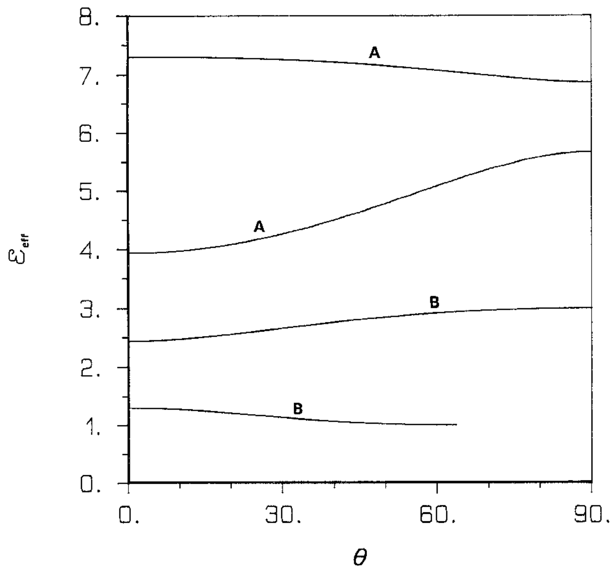


Fig. 4. ϵ_{eff} versus θ (in degrees) for the propagating hybrid surface modes of the slab in Fig. 3 with thickness $t = 0.15\lambda$ and a coplanar optical axis (Fig. 2). The A curves correspond to sapphire ($\epsilon_{\|} = 11.6$, $\epsilon_{\perp} = 9.4$). The B curves correspond to pyrolitic boron nitride ($\epsilon_{\|} = 3.40$, $\epsilon_{\perp} = 5.12$).

analysis of this waveguide: any approximate analysis neglecting the anisotropy could not predict this cutoff effect.

According to (50), $\phi_1(x' = \infty) = \phi_2(x' = \infty) = 0$ for any surface mode. Therefore, taking (48) into consideration, the proof of the symmetry of $\mathscr{L}$ in Appendix I still holds for surface modes. As a consequence, if $\mathbf{\Phi}_m$ and $\mathbf{\Phi}_n$ are any two surface modes, the orthogonality relation (29) is still valid, i.e.,

$$\int_0^\infty \left[\phi_{1m}(x')\phi_{1n}(x') + \frac{1}{\epsilon_\perp(x')}\phi_{2m}(x')\phi_{2n}(x') \right] dx' = \delta_{mn} \tag{56}$$

provided that a convenient normalization is chosen (Appendix II).

The surface modes, which are the proper eigenfunctions of $\mathscr{L}$, do not constitute a complete spectral representation for $\mathscr{L}$: there are pseudosurface modes [12] which do not belong to $D(\mathscr{L})$ but are, nevertheless, improper eigenfunctions of $\mathscr{L}$ [13].

The following discussion about the pseudosurface modes is only concerned with the case in which $\epsilon_{yz} \neq 0$. The transverse wavenumbers will be denoted by σ_s $(s = o, e)$ in the slab and by ρ in the air. The expressions for σ_s are the same as for h_s and

$$\rho^2 = 1 - \epsilon_{\text{eff}} \tag{57}$$

where ρ may take any positive value (continuous spectrum). For $0 < \rho < 1$ one has $0 < \epsilon_{\text{eff}} < 1$, i.e., the pseudosurface mode is propagating. For $\rho > 1$ one has $\epsilon_{\text{eff}} < 0$, i.e., the pseudosurface mode is evanescent.

For a given pseudosurface mode, if $0 < x' < t'$, then

$$\begin{aligned} \phi_1 &= A[\sin(\sigma_o x') + \Gamma \sin(\sigma_e x')] \\ \phi_2 &= A[\zeta_o \sigma_o \cos(\sigma_o x') + \Gamma \zeta_e \sigma_e \cos(\sigma_e x')] \end{aligned} \tag{58}$$

where ζ_s $(s = o, e)$ are the coupling coefficients with the same expressions as ξ_s. If $t' < x'$, then

$$\begin{aligned} \phi_1 &= \tau_1 A\{\cos[\rho(x' - t')] + \Lambda \sin[\rho(x' - t')]\} \\ \phi_2 &= \tau_2 A\{\cos[\rho(x' - t')] + \Lambda \sin[\rho(x' - t')]\} \end{aligned} \tag{59}$$

where ϕ_1 and ϕ_2 have the same standing-wave behavior in the air. In Appendix III expressions for coefficients τ_i $(i = 1, 2)$, Γ, and Λ are presented.

In Appendix IV it is shown, by means of a perturbation approach, that the following orthogonality relation is valid for the pseudosurface modes:

$$\int_0^\infty \left[\phi_1(x', \rho)\phi_1(x', \rho') + \frac{1}{\epsilon_\perp(x')}\phi_2(x', \rho)\phi_2(x', \rho') \right] dx' = \delta(\rho - \rho'). \tag{60}$$

According to (60), coefficient A in (58) and (59) must be properly normalized. Since $\delta(\rho + \rho') = 0$, one obtains from (60)

$$A = \sqrt{\frac{2}{\pi(\tau_1^2 + \tau_2^2)(1 + \Lambda^2)}}. \tag{61}$$

Owing to the behavior of the surface modes at infinity, one may conclude that the proof in Appendix I is still valid for a pair of surface and pseudosurface modes. Therefore, if $\mathbf{\Phi}_n(x')$ is any surface mode corresponding to a proper eigenvalue $\epsilon_{\text{eff}_n} > 1$ and $\mathbf{\Phi}(x', \rho)$ is any pseudosurface mode corresponding to an improper eigenvalue $\epsilon_{\text{eff}} < 1$, one has

$$\int_0^\infty \left[\phi_{1n}(x')\phi_1(x', \rho) + \frac{1}{\epsilon_\perp(x')}\phi_{2n}(x')\phi_2(x', \rho) \right] dx' = 0. \tag{62}$$

If the electromagnetic field of a uniaxial dielectric slab is characterized by $\mathbf{\Phi}(x') = [\phi_1(x'), \phi_1(x')]^T$, then

$$\mathbf{\Phi}(x') = \sum_{n=1}^{N} a_n \mathbf{\Phi}_n(x') + \int_0^\infty a(\rho)\mathbf{\Phi}(x', \rho)\, d\rho \tag{63}$$

where N is the total number of surface modes guided by the structure. According to (56), (60), and (62), one also obtains the following completeness relation from (63):

$$\int_0^\infty \left[\phi_1^2(x') + \frac{1}{\epsilon_\perp(x')}\phi_2^2(x') \right] dx' = \sum_{n=1}^{N} a_n^2 + \int_0^\infty a^2(\rho)\, d\rho. \tag{64}$$

Besides the theoretical interest of (63), this equation bears another feature which justifies its usefulness. In fact, in a building-block approach (e.g., in the analysis of abrupt discontinuities), it guarantees a rigorous description of the electromagnetic field in each subregion in terms of a known basis of two-vector transverse mode functions (discrete and continuous). Therefore, the present analytical formulation may be used as a first step in a mode matching procedure for the study of practical anisotropic wave-guiding structures.

One should finally note that the perturbation approach followed in deriving (60) is applicable to structures other than the slab waveguide of Fig. 3. In fact, this method can be easily applied as long as the anisotropic region is confined to a finite interval on x. For example, a uniaxial film with isotropic substrate and superstrate can be analyzed by this method.

V. Conclusions

Closed layered waveguides with lossless anisotropic layers in the polar configuration were analyzed through the unifying concept of a real self-adjoint operator. Layers such as magnetic crystals and gyrotropic media were considered. Orthogonality and completeness relations for the two-vector transverse eigenfunctions were derived. Special attention was paid to homogeneous layers although the theory is applicable to piecewise-continuous tensors $\bar{\epsilon}$ and $\bar{\mu}$ over a finite interval (regular problems).

As an extension of the former theory to open waveguides, a grounded uniaxial dielectric slab with a coplanar optical axis was analyzed. For this singular problem, defined over a semi-infinite interval, a complete spectral representation of the real self-adjoint operator—including the discrete and the continuous spectra—was presented. Finally, for regular problems, orthogonality and completeness relations for the two-vector eigenfunctions were also presented. This example of an open layered anisotropic waveguide is especially important if a building-block approach is used in the analysis of uniaxial dielectric planar structures for millimeter wave circuits. Moreover, as already pointed out, the present analysis can be easily applied to a class of multilayered waveguides used in integrated optics. The study of step discontinuities in these structures will be the subject of a forthcoming paper.

Appendix I
Proof of the Symmetry of $\mathscr{L}$

To prove the symmetry of $\mathscr{L}$, one has to prove that [13]

$$\Delta = \langle \mathscr{L}\cdot \boldsymbol{u},\boldsymbol{v}\rangle - \langle \boldsymbol{u},\mathscr{L}\cdot\boldsymbol{v}\rangle = 0 \tag{A1}$$

for $\boldsymbol{u},\boldsymbol{v} \in \mathrm{D}(\mathscr{L})$. After canceling the identical terms, one obtains

$$\Delta = \sum_{i=1}^{6} I_i \tag{A2}$$

with

$$I_1 = \int_0^{d'} \left[v_1 \partial_{x'}\left(\frac{1}{\mu_{zz}}\partial_{x'}u_1\right) - u_1\partial_{x'}\left(\frac{1}{\mu_{zz}}\partial_{x'}v_1\right)\right] dx'$$

$$I_2 = -\int_0^{d'} \left[b_1 v_1 \partial_{x'} u_2 + u_2 \partial_{x'}(b_1 v_1)\right] dx'$$

$$I_3 = \int_0^{d'} \left[v_1 \partial_{x'}(b_2 u_2) + b_2 u_2 \partial_{x'} v_1\right] dx'$$

$$I_4 = \int_0^{d'} \left[v_2 \partial_{x'}(b_1 u_1) + b_1 u_1 \partial_{x'} v_2\right] dx'$$

$$I_5 = -\int_0^{d'} \left[b_2 v_2 \partial_{x'} u_1 + u_1 \partial_{x'}(b_2 v_2)\right] dx'$$

$$I_6 = \int_0^{d'} \left[v_2 \partial_{x'}\left(\frac{1}{\epsilon_{zz}}\partial_{x'}u_2\right) - u_2\partial_{x'}\left(\frac{1}{\epsilon_{zz}}\partial_{x'}v_2\right)\right] dx'. \tag{A3}$$

The boundary conditions at $x' = 0, d'$ are (i) $u_1 = v_1 = \partial_{x'}u_2 = \partial_{x'}v_2 = 0$ for an electric wall and (ii) $u_2 = v_2 = \partial_{x'}u_1 = \partial_{x'}v_1 = 0$ for a magnetic wall. Then, using integration by parts together with (i) and/or (ii), one obtains, for $1 \leqslant i \leqslant 6$, $I_i = 0$. Hence, $\Delta = 0$ (q.e.d.).

Appendix II
Normalization of the Surface Modes

For a positive (negative) uniaxial crystal, i.e., for $\epsilon_{\|} > \epsilon_{\perp}$ ($\epsilon_{\perp} > \epsilon_{\|}$), one always has $h_e^2 > 0$ ($h_o^2 > 0$). However, if $\epsilon_{\|} > \epsilon_{\perp}$ ($\epsilon_{\perp} > \epsilon_{\|}$) there is a value $t' = t'_e$ ($t' = t'_o$) for which $h_o = 0$ ($h_e = 0$) and such that $h_o^2 > 0$ ($h_e^2 > 0$) for $t' < t'_e$ ($t' < t'_o$) and $h_o^2 < 0$ ($h_e^2 < 0$) for $t' > t'_e$ ($t' > t'_o$). Moreover, if $t' = t'_e$ ($t' = t'_o$) then $p_2 = 0$ ($p_1 = 0$) in (33). Therefore, for $t' = t'_s$ ($s = o, e$), (35)–(39) are no longer valid. For the sake of brevity, the normalization of the surface modes for these particular cases will be omitted. However, the determination of t'_s will be briefly presented below.

For a positive uniaxial crystal, if $t' = t'_e$, then $\epsilon_{\text{eff}} = \epsilon_{\perp}$ and

$$t'_e = \frac{1}{\sqrt{\epsilon_{\|} - \epsilon_{zz}}} \arctan\left(-\sqrt{\frac{\epsilon_{\|}-\epsilon_{zz}}{\epsilon_{\perp}-1}}\right). \tag{A4}$$

One should note that, in this particular case, $E_z = 0$.

For a negative uniaxial crystal, if $t' = t'_o$, then $\epsilon_{\text{eff}} = \epsilon_{\|}\epsilon_{\perp}/\epsilon_{zz}$ and t'_o is a solution of

$$\xi_e \gamma_o \delta'_e - \xi_o \gamma'_e \delta_o = 0 \tag{A5}$$

where

$$\gamma'_e = \lim_{h_e \to 0} \frac{\gamma_e}{h_e} = 1 + \alpha_a t'_o$$

$$\delta'_e = \lim_{h_e \to 0} \frac{\delta_e}{h_e} = \epsilon_{\perp}\alpha_a - h_o^2 t'_o. \tag{A6}$$

Coefficients A_e, B, and C of (49) and (50) can be expressed in terms of A_o in the form

$$A_e = aA_o \qquad B = bA_o \qquad C = cA_o. \tag{A7}$$

From (53) one obtains $a = -\gamma_o/\gamma_e$, and, from the continuity of ϕ_1 and ϕ_2 at $x' = t'$,

$$b = \sin(h_o t') + a\sin(h_e t')$$
$$c = \xi_o h_o \cos(h_o t') + a\xi_e h_e \cos(h_e t'). \tag{A8}$$

Taking into consideration (56), A_o is given by

$$A_o = \frac{1}{\sqrt{\eta - 2a\nu}} \tag{A9}$$

where

$$\eta = \eta_a + \eta_o + a^2\eta_e$$
$$\nu = \frac{1}{h_o}\cos(h_o t')\sin(h_e t') \tag{A10}$$

with ($s = o, e$)

$$\eta_a = \frac{b^2 + c^2}{2\alpha_a}$$

$$\eta_s = \frac{t'}{2}\left(1 + \frac{\xi_s^2 h_s^2}{\epsilon_{\perp}}\right) - \frac{\sin(2h_s t')}{4h_s}\left(1 - \frac{\xi_s^2 h_s^2}{\epsilon_{\perp}}\right). \tag{A11}$$

Appendix III
Coefficients of the Pseudosurface Modes

Following the same procedure as in Appendix II one obtains for the coefficients in (58) and (59)

$$\tau_1 = \sin(\sigma_o t') + \Gamma \sin(\sigma_e t')$$
$$\tau_2 = \zeta_o \sigma_o \cos(\sigma_o t') + \Gamma \zeta_e \sigma_e \cos(\sigma_e t') \quad \text{(A12)}$$

and

$$\Lambda = \frac{1}{\tau_1 \rho}\left[\sigma_o \cos(\sigma_o t') + \Gamma \sigma_e \cos(\sigma_e t')\right]. \quad \text{(A13)}$$

After substituting into (52) h_s and ξ_s with σ_s and ζ_s $(s = o, e)$, respectively, one also has

$$\Lambda = -\frac{\sigma_o^2}{\epsilon_\perp \tau_2 \rho}\left[\zeta_o \sin(\sigma_o t') + \Gamma \zeta_e \sin(\sigma_e t')\right]. \quad \text{(A14)}$$

Hence, from (A13) and (A14), Γ may be evaluated according to the algebraic equation

$$g_2 \Gamma^2 + g_1 \Gamma + g_0 = 0 \quad \text{(A15)}$$

with

$$g_0 = \sigma_o^2 \zeta_o \left[\sin^2(\sigma_o t') + \epsilon_\perp \cos^2(\sigma_o t')\right]$$
$$g_1 = (\zeta_o + \zeta_e)\left[\sigma_o^2 \sin(\sigma_o t') \sin(\sigma_e t') + \epsilon_\perp \sigma_o \sigma_e \cos(\sigma_o t') \cos(\sigma_e t')\right]$$
$$g_2 = \zeta_e\left[\sigma_o^2 \sin^2(\sigma_e t') + \epsilon_\perp \sigma_e^2 \cos^2(\sigma_e t')\right]. \quad \text{(A16)}$$

As $g_0 g_2$ and $\zeta_o \zeta_e$ have the same sign and since $\zeta_o \zeta_e = -\epsilon_\perp / \sigma_o^2 < 0$ then $\Gamma^2 > 0$, i.e., Γ is always real. Hence, according to (A12) and (A13), τ_1, τ_2, and Λ are also real.

Appendix IV
Orthogonality Relation for the Pseudosurface Modes

The orthogonality relation for the pseudosurface modes may be derived following a perturbation approach similar to the one presented by Sammut [10]. In fact, the operator $\overline{\mathscr{L}}$ in Section IV may be split into

$$\overline{\mathscr{L}} = \overline{\mathscr{L}}_a + \overline{\mathscr{M}} \quad \text{(A17)}$$

where $\overline{\mathscr{M}}$ is a "small" perturbation of an operator $\overline{\mathscr{L}}_a$. For the case under consideration, it is appropriate to define an operator $\overline{\mathscr{L}}_a$ equivalent to $\overline{\mathscr{L}}$ for $t' < x'$, i.e.,

$$\overline{\mathscr{L}}_a = \begin{bmatrix} \partial_{x'}^2 + 1 & 0 \\ 0 & \partial_{x'}^2 + 1 \end{bmatrix} \quad \text{(A18)}$$

and hence

$$\overline{\mathscr{M}} = H(t' - x') \begin{bmatrix} \dfrac{\epsilon_\parallel \epsilon_\perp}{\epsilon_{zz}} - 1 & -\dfrac{\epsilon_{yz}}{\epsilon_{zz}} \partial_{x'} \\ \partial_{x'} \dfrac{\epsilon_{yz}}{\epsilon_{zz}} & \partial_{x'} \dfrac{1}{\epsilon_{zz}} \partial_{x'} - \partial_{x'}^2 \end{bmatrix} \quad \text{(A19)}$$

where

$$H(t' - x') = \begin{cases} 1 & \text{for } x' < t' \\ 0 & \text{for } t' < x'. \end{cases} \quad \text{(A20)}$$

If $\mathbf{\Phi}_a$ is an improper eigenfunction of $\overline{\mathscr{L}}_a$, then

$$(\overline{\mathscr{L}}_a - \epsilon_{\text{eff}} \overline{\mathscr{W}}) \cdot \mathbf{\Phi}_a = \mathbf{0} \quad \text{(A21)}$$

whereas, according to (19),

$$(\overline{\mathscr{L}} - \epsilon_{\text{eff}} \overline{\mathscr{W}}) \cdot \mathbf{\Phi} = \mathbf{0} \quad \text{(A22)}$$

for an improper eigenfunction of $\overline{\mathscr{L}}$. From (A17), (A21), and (A22) one may write [10], [13]

$$\mathbf{\Phi} = \mathbf{\Phi}_a - \lim_{\delta \to 0} (\overline{\mathscr{L}}_a - \epsilon_{\text{eff}} \overline{\mathscr{W}} - j\delta \overline{\mathscr{M}})^{-1} \cdot \overline{\mathscr{M}} \cdot \mathbf{\Phi} \quad \text{(A23)}$$

and

$$\langle \mathbf{\Phi}(x', \rho), \overline{\mathscr{W}} \cdot \mathbf{\Phi}(x', \rho') \rangle = \langle \mathbf{\Phi}_a(x', \rho), \overline{\mathscr{W}} \cdot \mathbf{\Phi}_a(x', \rho') \rangle. \quad \text{(A24)}$$

However, according to (A18), the eigenfunctions of $\overline{\mathscr{L}}_a$ are uncoupled TE and TM modes in the half-space. Therefore, as is well known, one may write, if an appropriate normalization is made for both TE and TM modes,

$$\langle \mathbf{\Phi}_a(x', \rho), \overline{\mathscr{W}} \cdot \mathbf{\Phi}_a(x', \rho') \rangle = \delta(\rho - \rho') \quad \text{(A25)}$$

where ρ was introduced in (57).

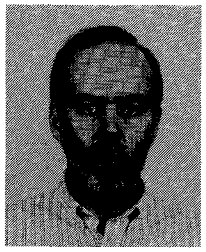
From (A24) and (A25) the orthogonality relation (60) is found.

References

[1] D. W. Berreman, "Optics in stratified and anisotropic media: 4×4 matrix formulation," *J. Opt. Soc. Amer.*, vol. 62, pp. 502–510, Apr. 1972.
[2] A. A. Mostafa, C. M. Krowne, and K. A. Zaki, "Numerical spectral matrix method for propagation in general layered media: Application to isotropic and anisotropic substrates," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, pp. 1399–1407, Dec. 1987.
[3] A. Knoesen, T. K. Gaylord, and M. G. Moharam, "Hybrid guided modes in uniaxial dielectric planar waveguides," *J. Lightwave Technol.*, vol. 6, pp. 1083–1104, June 1988.
[4] O. Schwelb, "Stratified lossy anisotropic media: General characteristics," *J. Opt. Soc. Amer. A*, vol. 3, pp. 188–193, Feb. 1986.
[5] S. K. Chung and S. S. Kim, "An exact wave-optics analysis of optical waveguide with anisotropic and gyrotropic materials," *J. Appl. Phys.*, vol. 63, pp. 5654–5659, June 1988.
[6] E. El-Sharawy and R. W. Jackson, "Coplanar waveguide and slot line on magnetic substrates: Analysis and experiment," *IEEE Trans. Microwave Theory Tech.*, vol. 36, pp. 1071–1079, June 1988.
[7] C. Paiva and A. M. Barbosa, "An analytical approach to stratified waveguides with anisotropic layers in the longitudinal or polar configurations," *J. Electromagn. Waves and Appl.*, vol. 4, pp. 75–93, 1990.
[8] S. T. Peng and A. A. Oliner, "Guidance and leakage properties of a class of open dielectric waveguides: Part I—Mathematical formulation," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-29, pp. 843–855, Sept. 1981.
[9] A. D. Bresler, G. H. Joshi, and N. Marcuvitz, "Orthogonality properties for modes in passive and active uniform wave guides," *J. Appl. Phys.*, vol. 29, pp. 794–799, May 1958.
[10] R. A. Sammut, "Orthogonality and normalization of radiation modes in dielectric waveguides," *J. Opt. Soc. Amer.*, vol. 72, pp. 1335–1337, Oct. 1982.
[11] M. Mrozowski and J. Mazur, "General analysis of a parallel-plate inhomogeneously filled with gyromagnetic media," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-34, pp. 388–395, Apr. 1986.
[12] V. V. Shevchenko, *Continuous Transitions in Open Waveguides.* Boulder, CO: Golem Press, 1971, ch. 2.

[13] B. Friedman, *Principles and Techniques of Applied Mathematics*. New York: Wiley, 1956, ch. 4.
[14] H. C. Chen, *Theory of Electromagnetic Waves: A Coordinate-Free Approach*. New York: McGraw-Hill, 1985, pp. 71, 216, 220.
[15] A. Yariv and P. Yeh, *Optical Waves in Crystals*. New York: Wiley, 1984, pp. 94–104.
[16] L. Landau and E. Lifchitz, *Physique Statistique*. Moscow: MIR, 1967, pp. 452–457.

Carlos R. Paiva was born in Lisbon, Portugal, on December 12, 1957. He received the Licenciatura and master's degrees in electrical engineering from the Instituto Superior Técnico (IST)—Technical University of Lisbon, in 1982 and 1986, respectively. He is currently studying for the Ph.D. degree in the Department of Electrical Engineering at IST. His research interests are in the areas of electromagnetic wave propagation in anisotropic structures and passive devices for millimeter-wave and optical integrated circuits.



Afonso M. Barbosa (S'80–A'83) was born in Coimbra, Portugal, on May 27, 1950. He received the Licenciatura degree in Electrical Engineering from the Instituto Superior Ténico (IST), Lisbon, Portugal, in 1972, the master's degree in electronic engineering from NUFFIC, Netherlands, in 1974, and the doctoral degree in electrical engineering from IST in 1984.

He is currently an Associate Professor of Microwaves in the Department of Electrical and Computer Engineering, IST. His current interests are in electromagnetic wave theory and such applications as microwave, millimeter-wave, and optical waveguide structures, antennas, and scattering.